Integrability theorems for Jacobi series and Parseval's formulae

By YOSHIMITSU HASEGAWA in Hirosaki (Japan)

1. Introduction and statement of main results

Let $P_n^{(\alpha,\beta)}(x)$ be the Jacobi polynomial of degree n and order (α, β), defined by

$$(1-x)^\alpha(1+x)^\beta P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}\frac{d^n}{dx^n}\{(1-x)^{\alpha+n}(1+x)^{\beta+n}\}, \qquad \alpha, \beta > -1.$$

These plynomials are orthogonal on $(-1, 1)$ with respect to the weight function $(1-x)^\alpha(1+x)^\beta$ and normalized by

$$P_n^{(\alpha,\beta)}(1) = \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)\Gamma(n+1)}.$$

For convenience we often set $x=\cos\theta$. The functions $P_n^{(\alpha,\beta)}(\cos\theta)$ are orthogonal on $(0, \pi)$ with respect to weight function

$$\varrho^{(\alpha,\beta)}(\theta) = \left(\sin\frac{\theta}{2}\right)^{2\alpha+1}\left(\cos\frac{\theta}{2}\right)^{2\beta+1}$$

and satisfy

$$(1.2)\ \frac{1}{\omega_n^{(\alpha,\beta)}} = \int_0^\pi \{P_n^{(\alpha,\beta)}(\cos\theta)\}^2\varrho^{(\alpha,\beta)}(\theta)\,d\theta = \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}$$

(notice $\omega_0^{(-1/2,-1/2)} = 1/\pi$).

A Jacobi series is of the form

$$(1.3) \qquad \sum_{n=0}^{\infty} a_n\omega_n^{(\alpha,\beta)}P_n^{(\alpha,\beta)}(\cos\theta),$$

where a_n $(n=0, 1, 2, \ldots)$ are real numbers. If $f(\theta)$ is Lebesgue-integrable on $(0, \pi)$ with respect to the weight function $\varrho^{(\alpha,\beta)}(\theta)$, then we write $f(\theta)\in L([0,\pi]; \alpha, \beta)$ and denote its Fourier—Jacobi series by

$$(1.4) \qquad f(\theta) \sim \sum_{n=0}^{\infty} a_n\omega_n^{(\alpha,\beta)}P_n^{(\alpha,\beta)}(\cos\theta),$$

where the Fourier—Jacobi coefficients are given by

$$a_n = \int_0^{\pi} f(\theta) P_n^{(\alpha,\beta)}(\cos\theta)\varrho^{(\alpha,\beta)}(\theta)\,d\theta.$$

Definition 1. A function $f(\theta)$ is said to be *R-integrable on* $(0, \pi)$ *with respect to the weight function* $\varrho^{(\alpha,\beta)}(\theta)$ if $f(\theta)$ is Lebesgue-integrable on any closed interval $[\theta_1, \theta_2]$, $0<\theta_1<\theta_2<\pi$, and if

$$\lim_{\substack{\theta_1\to+0\\ \theta_2\to\pi-0}} \int_{\theta_1}^{\theta_2} f(\theta)\varrho^{(\alpha,\beta)}(\theta)\,d\theta = \int_{\to 0}^{\to\pi} f(\theta)\varrho^{(\alpha,\beta)}(\theta)\,d\theta$$

exists and is finite. Then we write $f(\theta)\in R([0, \pi]; \alpha, \beta)$.

Definition 2. A function $\varphi(u)$ is said to be *slowly varying* if $\varphi(u)$ is positive and continuous in $u\geqq 0$, and if

$$\frac{\varphi(tu)}{\varphi(u)} \to 1 \quad \text{as} \quad u\to\infty$$

for every $t>0$.

A slowly varying function $\varphi(u)$ has the following properties.

(S1) $\dfrac{\varphi(tu)}{\varphi(u)} \to 1$ as $u\to\infty$ uniformly for $0<T_1\leqq t\leqq T_2<\infty$,

where T_1 and T_2 are any two fixed values.

(S2) $u^{\gamma}\varphi(u)\to\infty$, $u^{-\gamma}\varphi(u)\to 0$ as $u\to\infty$ for every $\gamma>0$.

(S3) If we set

$$\varphi_1(u) = u^{-\gamma}\sup_{0\leqq v\leqq u}\{v^{\gamma}\varphi(v)\},\quad \varphi_2(u) = u^{\gamma}\sup_{u\leqq v<\infty}\{v^{-\gamma}\varphi(v)\}\quad \text{for}\quad \gamma>0,$$

then $\varphi_m(u)/\varphi(u)\to 1$ as $u\to\infty$ $(m=1, 2)$. Furthermore, $u^{\gamma}\varphi_1(u)$ is non-decreasing, and $u^{-\gamma}\varphi_2(u)$ is non-increasing.

(S4) For $\gamma>0$, we have

$$\varphi(tu)\leqq A' t^{-\gamma}\varphi(u) \quad\text{and}\quad \varphi(u/t)\leqq A'' t^{-\gamma}\varphi(u)\quad \text{for every}\quad u\geqq 0,\ 1\geqq t>0,$$

where A' and A'' are positive constants depending only on γ and φ.

(S1), (S2) and (S4) are due to S. Igari [7], and (S3) is due to S. Aljančić, R. Bojanić and M. Tomić [1].

C. C. Ganser [6] gave some results with respect to L-integrability of ultraspherical series. First we give a sufficient condition concerning coefficients in order that the Jacobi series (1.3) converges to a function in $0<\theta<\pi$, and then prove five results (Theorems 1, 2 and 3) with respect to R- or L-integrability of the function.

Theorem 1. *Let $\alpha, \beta \geqq -1/2$. Suppose that the Jacobi series (1.3) satisfies the conditions:*

(J1) *for* $\alpha > -1/2$,

$$\sum_{n=1}^{\infty} n^{1/2}|\Delta a_n| < \infty \quad \textit{and} \quad n^{1/2}a_n \to 0 \quad \textit{as} \quad n \to \infty \quad (\Delta a_n = a_n - a_{n+1}),$$

(J2) *for* $\alpha = -1/2$,

$$\sum_{n=1}^{\infty} |\Delta(n^{1/2}a_n)| < \infty \quad \textit{and} \quad n^{1/2}a_n \to 0 \quad \textit{as} \quad n \to \infty.$$

Then the Jacobi series (1.3) converges to a function $f(\theta)$ in $0<\theta<\pi$, uniformly in $\varepsilon \leqq \theta \leqq \pi-\varepsilon$ for every $0<\varepsilon<\pi$. Moreover we have:

(i) *If* $\alpha, \beta \geqq -1/2$, *then*

$$f(\theta)\theta^{\mu-\alpha-1/2}(\pi-\theta)^{\nu-\beta-3/2} \in L([0, \pi]; \alpha, \beta) \quad \textit{for any} \quad \mu, \nu > 0.$$

(ii) *If* $\alpha \geqq -1/2$ *and* $\beta > -1/2$, *then*

$$f(\theta)\theta^{-\alpha-1/2}(\pi-\theta)^{-\beta-3/2} \in R([0, \pi]; \alpha, \beta).$$

Remark 1. As it is easily seen from the proof of Theorem 1, we have under the conditions of Theorem 1 that

$$\int_{\to 0}^{\pi} f(\theta)\theta^{-\alpha-1/2}(\pi-\theta)^{\nu-\beta-3/2}\varrho^{(\alpha,\beta)}(\theta)\, d\theta$$

exists and is finite if $\alpha, \beta \geqq -1/2$ and $\nu>0$. The particular case $\alpha=\beta=-1/2$ and $\nu=1$ (Fourier cosine series) is stated by BARY [4; p. 209—211].

Remark 2. Using (3.3) for $\tau=0$, the conditions in (J1) imply $\sum_{n=1}^{\infty} n^{-1/2}|a_n| < \infty$. Hence, by $\Delta(n^{1/2}a_n) = n^{1/2}\Delta a_n + a_{n+1}\Delta n^{1/2}$, the conditions in (J1) imply the conditions in (J2). Conversely, it is clear that the conditions

$$\sum_{n=1}^{\infty} |\Delta(n^{1/2}a_n)| < \infty, \quad \sum_{n=1}^{\infty} n^{-1/2}|a_n| < \infty \quad \text{and} \quad n^{1/2}a_n \to 0 \quad \text{as} \quad n \to \infty$$

imply the conditions in (J1).

Theorem 2. *Let $\alpha \geqq -1/2$ and $\beta > -1/2$. Suppose that the Jacobi series (1.3) satisfies the conditions:*

(J3) *for* $\alpha > -1/2$,

$$\sum_{n=1}^{\infty} |\Delta a_n| n^{1/2} \log(n+1) < \infty \quad \textit{and} \quad n^{1/2}a_n \to 0,$$

(J4) *for* $\alpha=-1/2$,

$$\sum_{n=1}^{\infty}|\Delta(n^{1/2}a_n)|\log(n+1)<\infty \quad \text{and} \quad n^{1/2}a_n\to 0.$$

Then the Jacobi series (1.3) *convergens to a function* $f(\theta)$ *in* $0<\theta<\pi$, *and furthermore,*

$$f(\theta)\theta^{-\alpha-1/2}(\pi-\theta)^{-\beta-3/2}\in L([0,\pi];\alpha,\beta).$$

Theorem 3. *Let* $\alpha,\beta\geqq -1/2$ *and* $0<\delta<1$, *and let* $\varphi(u)$ *be a slowly varying function. Suppose that* $\{a_n\}$ $(n=0,1,2,\dots)$ *is a non-negative sequence such that* $\{n^{1/2}a_n\}$ $(n=1,2,\dots)$ *is non-increasing and tends to zero as* $n\to\infty$, *and that* $\sum_{n=1}^{\infty}n^{-\delta+1/2}\varphi(n)a_n$ *converges. Then the Jacobi series* (1.3) *converges to a function* $f(\theta)$ *in* $0<\theta<\pi$. *Moreover,*

(i) *if* $-1/2\leqq\beta<1/2$, $\nu\geqq\delta-1$ *and* $\nu>-\beta-1/2$, *then*

$$f(\theta)\theta^{\delta-\alpha-3/2}(\pi-\theta)^{\nu-\beta-3/2}\varphi\left(\frac{1}{\theta(\pi-\theta)}\right)\in L([0,\pi];\alpha,\beta);$$

(ii) *if* $\beta\geqq 1/2$, *then*

$$f(\theta)\theta^{\delta-\alpha-3/2}(\pi-\theta)^{\delta-\beta-5/2}\varphi\left(\frac{1}{\theta(\pi-\theta)}\right)\in L([0,\pi];\alpha,\beta).$$

The next theorems are extensions of theorems of M. and S. IZUMI [8] on the integrability of trigonometric series.

Theorem 4. *Let* $\alpha,\beta\geqq -1/2$, *and let*

$$f(\theta)\in L([0,\pi];\alpha,\beta),\quad g(\theta)\in L\left([0,\pi];\frac{1}{2}\left(\alpha-\frac{1}{2}\right),\frac{1}{2}\left(\beta-\frac{1}{2}\right)\right).$$

Further let

$$f(\theta)\sim\sum_{n=0}^{\infty}a_n\omega_n^{(\alpha,\beta)}P_n^{(\alpha,\beta)}(\cos\theta),\quad g(\theta)\sim\sum_{n=0}^{\infty}b_n\omega_n^{(\alpha,\beta)}P_n^{(\alpha,\beta)}(\cos\theta).$$

Suppose that $g(\theta)$ *is non-negative on* $(0,\pi)$, *that* $\theta^{\alpha+1/2}g(\theta)$ *is non-increasing on* $(0,\pi/2]$, *and that* $(\pi-\theta)^{\beta+1/2}g(\theta)$ *is non-decreasing on* $(\pi/2,\pi)$. *If the series*

$$\sum_{n=1}^{\infty}n^{1/2}a_n\left(\int_0^{1/n}\theta^{\alpha+1/2}g(\theta)\,d\theta+\int_0^{1/n}\theta^{\beta+1/2}g(\pi-\theta)\,d\theta\right)$$

converges absolutely, then $f(\theta)g(\theta)\in R([0,\pi];\alpha,\beta)$ *and Parseval's formula*

$$\sum_{n=0}^{\infty}a_nb_n\omega_n^{(\alpha,\beta)}=\int_{\to 0}^{\to\pi}f(\theta)g(\theta)\varrho^{(\alpha,\beta)}(\theta)\,d\theta$$

holds, where the series on the left hand side converges absolutely.

Corollary 1. *Let* $\alpha, \beta \geqq -1/2$ *and* $0<\delta<1$. *Let* $f(\theta)\in L([0, \pi]; \alpha, \beta)$ *and let its Fourier—Jacobi series be* (1.4). *Suppose that* $\varphi(u)$ *is a slowly varying function such that* $u^{\gamma}\varphi(u)$ *is non-decreasing on* $(0, \infty)$ *for every* $\gamma>0$. *If the series* $\sum_{n=1}^{\infty} n^{-\delta+1/2}\varphi(n)a_n$ *converges absolutely, then*

$$f(\theta)\theta^{\delta-\alpha-3/2}(\pi-\theta)^{\delta-\beta-3/2}\varphi\left(\frac{1}{\theta(\pi-\theta)}\right)\in R([0, \pi]; \alpha, \beta).$$

Theorem 5. *Let* $\alpha, \beta \geqq -1/2$. *Suppose that the Jacobi series*

$$\sum_{n=0}^{\infty} b_n \omega_n^{(\alpha,\beta)} P_n^{(\alpha,\beta)}(\cos\theta) \tag{1.5}$$

satisfies the following conditions:

(J5) $b_0 \geqq 0$,

(J6) $\{n^{1/2}b_n\}$ $(n = 1, 2, \ldots)$ *is a non-increasing sequence converging to zero,*

(J7) $\sum_{n=1}^{\infty} n^{-1/2}b_n$ *converges when* $\alpha > -1/2$.

Let

$$G(\theta) = \begin{cases} \sum_{n=0}^{[1/\theta]} (n+1)^{\alpha+1}b_n & \text{for} \quad 0 \leqq \theta \leqq \dfrac{\pi}{2} \\ \sum_{n=0}^{[1/(\pi-\theta)]} (n+1)^{\beta}b_n & \text{for} \quad \dfrac{\pi}{2} < \theta \leqq \pi, \end{cases} \tag{1.6}$$

where $[u]$ *denotes the largest integer* $\leqq u$. *Moreover, let* $f(\theta)\in L([0, \pi]; \alpha, \beta)$ *and let its Fourier—Jacobi series be* (1.4). *Then the series* (1.5) *converges to a function* $g(\theta)$ *in* $0<\theta<\pi$. *Moreover, if* $f(\theta)G(\theta)\in L([0, \pi]; \alpha, \beta)$, *then the series* $\sum_{n=1}^{\infty} a_n b_n \omega_n^{(\alpha,\beta)}$ *converges and Parseval's formula*

$$\int_0^{\pi} f(\theta)g(\theta)\varrho^{(\alpha,\beta)}(\theta)\,d\theta = \sum_{n=0}^{\infty} a_n b_n \omega_n^{(\alpha,\beta)}$$

holds, the integral converging absolutely.

Theorem 5 has the following two corollaries.

Corollary 2. *Let* $\beta \geqq \alpha > -1/2$ *and* $0<\delta<\alpha+1/2$. *Let* $f(\theta)\in L([0, \pi]; \alpha, \beta)$ *and let its Fourier—Jacobi series be* (1.4). *Suppose that* $\varphi(u)$ *is a slowly varying function such that* $u^{-\gamma}\varphi(u)$ *is non-increasing on* $(0, \infty)$ *for every* $\gamma>0$. *If*

$$f(\theta)\theta^{\delta-\alpha-3/2}(\pi-\theta)^{\delta-\beta-1/2}\varphi\left(\frac{1}{\theta(\pi-\theta)}\right)\in L([0, \pi]; \alpha, \beta),$$

then the series $\sum_{n=1}^{\infty} n^{-\delta+1/2}\varphi(n)a_n$ *converges.*

Corollary 3. *Let* $\alpha \geqq \beta \geqq -1/2$ *and* $\alpha+1/2<\delta<\alpha+3/2$. *Let* $f(\theta)\in L([0,\pi];\alpha,\beta)$ *and let its Fourier—Jacobi series be* (1.4). *Let* $\varphi(u)$ *be defined as in Corollary 2. If*

$$f(\theta)\theta^{\delta-\alpha-3/2}\varphi\left(\frac{1}{\theta}\right)\in L([0,\pi];\alpha,\beta),$$

then the series $\sum_{n=1}^{\infty} n^{-\delta+1/2}\varphi(n)a_n$ *converges.*

Throughout the paper, the letter K, with or without a sufix, denotes a positive constant, not necessarily the same on each appearance.

2. Preliminary estimates for Jacobi polynomials

Using Stirling's formula (see [9; p. 32]),

$$\Gamma(u) = \sqrt{2\pi}u^{u-1/2}e^{-u}\left\{1+\frac{1}{12u}+O\left(\frac{1}{u^2}\right)\right\} \quad \text{as} \quad u\to\infty,$$

we have from (1.1) and (1.2), respectively,

$$P_n^{(\alpha,\beta)}(1) = \frac{n^\alpha}{\Gamma(\alpha+1)}\left\{1+\frac{A}{n}+O\left(\frac{1}{n^2}\right)\right\} \quad \text{as} \quad n\to\infty, \tag{2.1}$$

where A is a constant depending only on α, and

$$\omega_n^{(\alpha,\beta)} = 2n\left\{1+\frac{B}{n}+O\left(\frac{1}{n^2}\right)\right\} \quad \text{as} \quad n\to\infty, \tag{2.2}$$

where B is a constant depending only on α and β. Furthermore, by [11; (4.1.3), (4.5.3), (4.5.4)],

$$P_n^{(\alpha,\beta)}(x) = (-1)^n P_n^{(\beta,\alpha)}(-x), \tag{2.3}$$

$$\sum_{k=0}^{n}\omega_k^{(\alpha,\beta)}P_k^{(\alpha,\beta)}(1)P_k^{(\alpha,\beta)}(x) = \frac{\alpha+1}{2n+\alpha+\beta+2}\omega_n^{(\alpha+1,\beta)}P_n^{(\alpha+1,\beta)}(1)P_n^{(\alpha+1,\beta)}(x), \tag{2.4}$$

$$\left(n+\frac{\alpha+\beta+2}{2}\right)(1-x)P_n^{(\alpha+1,\beta)}(x) = (n+\alpha+1)P_n^{(\alpha,\beta)}(x)-(n+1)P_{n+1}^{(\alpha,\beta)}(x). \tag{2.5}$$

By (2.2), (2.3) and [12; (7.32.5)], we have

$$P_n^{(\alpha,\beta)}(\cos\theta) = \begin{cases} \theta^{-\alpha-1/2}(\pi-\theta)^{-\beta-1/2}O(n^{-1/2}) & \text{for } cn^{-1}\leqq\theta\leqq\pi-cn^{-1} \text{ and } \alpha,\beta\geqq-\dfrac{1}{2}, \\ O(n^\alpha) & \text{for } 0\leqq\theta\leqq\dfrac{\pi}{2} \text{ and } \alpha\geqq-\dfrac{1}{2}, \\ O(n^\beta) & \text{for } \dfrac{\pi}{2}<\theta\leqq\pi \text{ and } \beta\geqq-\dfrac{1}{2} \end{cases} \tag{2.6}$$

as $n\to\infty$, where c is a positive constant depending only on α and β. Also,

$$\sqrt{\omega_n^{(\alpha,\beta)}}\,P_n^{(\alpha,\beta)}(\cos\theta)\left(\sin\frac{\theta}{2}\right)^{\alpha+1/2}\left(\cos\frac{\theta}{2}\right)^{\beta+1/2}=$$

(2.7)
$$= D\cdot\cos\left\{\left(n+\frac{\alpha+\beta+1}{2}\right)\theta-\frac{\pi}{2}\left(\alpha+\frac{1}{2}\right)\right\}+(\sin\theta)^{-1}O(n^{-1})$$

for $cn^{-1}\leqq\theta\leqq\pi-cn^{-1}$ and $\alpha,\beta>-1$, as $n\to\infty$,

[11; (8.21.18)], and more exatly,

$$\sqrt{\omega_n^{(\alpha,\beta)}}\,P_n^{(\alpha,\beta)}(\cos\theta)\left(\sin\frac{\theta}{2}\right)^{\alpha+1/2}\left(\cos\frac{\theta}{2}\right)^{\beta+1/2}=$$

(2.8)
$$=(D+D^*n^{-1})\cos\left\{\left(n+\frac{\alpha+\beta+1}{2}\right)\theta-\frac{\pi}{2}\left(\alpha+\frac{1}{2}\right)\right\}+$$

$$+\left(En^{-1}\cot\frac{\theta}{2}+E^*n^{-1}\tan\frac{\theta}{2}\right)\sin\left\{\left(n+\frac{\alpha+\beta+1}{2}\right)\theta-\frac{\pi}{2}\left(\alpha+\frac{1}{2}\right)\right\}+(\sin\theta)^{-2}O(n^{-2})$$

for $cn^{-1}\leqq\theta\leqq\pi-cn^{-1}$ and $\alpha,\beta>-1$, as $n\to\infty$,

[3; p. 585], where D, D^*, E and E^* are constants depending only on α and β. Finally,

(2.9)
$$\int_0^1(1-x)^\mu|P_n^{(\alpha,\beta)}(x)|\,dx=\begin{cases}O(n^{-1/2}\log n) & \text{for } 2\mu=\alpha-\frac{3}{2} \text{ and } \alpha,\beta,\mu>-1,\\ O(n^{-1/2}) & \text{for } 2\mu>\alpha-\frac{3}{2} \text{ and } \alpha,\beta,\mu>-1\end{cases}$$

as $n\to\infty$ [11; (7.34.1)].

3. Proofs of Theorems 1 and 2

For the proof of Theorem 1, we need the following lemma.

Lemma 1. *Let $\alpha,\beta\geqq-1/2$. Suppose that a sequence $\{a_n\}$ of real numbers satisfies the condition*

(3.1)
$$\sum_{n=1}^{\infty}n^{\alpha+1/2}|\Delta d_n|<\infty \quad \textit{and} \quad n^{1/2}a_n\to 0 \quad \textit{as} \quad n\to\infty,$$

where $d_n=a_n/P_n^{(\alpha,\beta)}(1)$. Then condition (3.1) *is equivalent to conditions* (J1) *and* (J2).

Proof. By (2.1), we have

$$\Delta d_n=\Gamma(\alpha+1)\left\{a_nn^{-\alpha}\left(1-\frac{A}{n}+O\left(\frac{1}{n^2}\right)\right)-a_{n+1}(n+1)^{-\alpha}\left(1-\frac{A}{n+1}+O\left(\frac{1}{n^2}\right)\right)\right\}=$$

$$=\Gamma(\alpha+1)\left\{\left(1-\frac{A}{n}\right)\Delta(n^{-\alpha}a_n)+a_n\cdot O(n^{-\alpha-2})-a_{n+1}\cdot O((n+1)^{-\alpha-2})\right\}$$

as $n\to\infty$. Since $n^{1/2}a_n\to 0$ as $n\to\infty$, we have

$$\sum_{n=1}^{\infty}(|a_n|n^{-\alpha-2})n^{\alpha+1/2}=\sum_{n=1}^{\infty}n^{-2}\cdot n^{1/2}|a_n|<\infty.$$

Hence

$$\left|\sum_{m=1}^{n}m^{\alpha+1/2}|\Delta d_m|-\Gamma(\alpha+1)\sum_{m=1}^{n}m^{\alpha+1/2}\left|\left(1-\frac{A}{m}\right)\Delta(m^{-\alpha}a_m)\right|\right|\leqq$$

$$\leqq\sum_{m=1}^{n}m^{\alpha+1/2}\left|\Delta d_m-\Gamma(\alpha+1)m^{\alpha+1/2}\left(1-\frac{A}{m}\right)\Delta(m^{-\alpha}a_m)\right|\leqq$$

$$\leqq K\sum_{m=1}^{\infty}m^{\alpha+1/2}(|a_m|m^{-\alpha-2})<\infty.$$

Thus (3.1) is clearly equivalent to the condition

$$\sum_{n=1}^{\infty}n^{\alpha+1/2}|\Delta(n^{-\alpha}a_n)|<\infty \quad\text{and}\quad n^{1/2}a_n\to 0 \quad\text{as}\quad n\to\infty. \tag{3.2}$$

Now, since (J2) coincides with (3.2) for $\alpha=-1/2$, it is enough to show that (J1) is equivalent to (3.2) for $\alpha>-1/2$.

Let $\tau>-1/2$. By Abel's transformation, we have, for $n=2, 3, \dots,$

$$\begin{aligned}\sum_{m=1}^{n}m^{-1/2}|a_m|&=\sum_{m=1}^{n-1}\left(\sum_{s=1}^{m}s^{\tau-1/2}\right)\Delta(m^{-\tau}|a_m|)+\left(\sum_{s=1}^{n}s^{\tau-1/2}\right)n^{-\tau}|a_n|\leqq\\ &\leqq K_1\sum_{m=1}^{n-1}m^{\tau+1/2}|\Delta(m^{-\tau}a_m)|+K_2n^{1/2}|a_n|.\end{aligned}\tag{3.3}$$

If we assume (J1) or (3.2) for $\alpha>-1/2$, then we get $\sum_{n=1}^{\infty}n^{-1/2}|a_n|<\infty$, using (3.3) for $\tau=0$ or α. By $\Delta(n^{-\alpha}a_n)=n^{-\alpha}\Delta a_n+a_{n+1}\Delta n^{\alpha}$, we get

$$\left|\sum_{m=1}^{n}m^{\alpha+1/2}|\Delta(m^{-\alpha}a_m)|-\sum_{m=1}^{n}m^{1/2}|\Delta a_m|\right|\leqq\sum_{m=1}^{n}|m^{\alpha+1/2}\Delta(m^{-\alpha}a_m)-m^{1/2}\Delta a_m|\leqq$$

$$\leqq\sum_{m=1}^{n}m^{\alpha+1/2}|a_{m+1}\Delta m^{-\alpha}|\leqq K\sum_{m=1}^{n}m^{-1/2}|a_m|.$$

Hence, (J1) is clearly equivalent to (3.2) for $\alpha>-1/2$. Thus Lemma 1 is proved.

Proof of Theorem 1. Let $d_n=a_n/P_n^{(\alpha,\beta)}(1)$. By Lemma 1, conditions (J1) and (J2) are equivalent to (3.1). We put $0<\theta<\pi$. From (2.4) and Abel's transformation, we have, for any m, n $(n>m\geqq 0)$,

$$\sum_{s=m+1}^{n} a_s \omega_s^{(\alpha,\beta)} P_s^{(\alpha,\beta)}(\cos\theta) =$$

$$\begin{aligned}
&= \sum_{s=m+1}^{n} (\Delta d_s) \frac{\alpha+1}{2s+\alpha+\beta+2} \omega_s^{(\alpha+1,\beta)} P_s^{(\alpha+1,\beta)}(1) P_s^{(\alpha+1,\beta)}(\cos\theta) - \\
&\quad - d_{m+1} \cdot \frac{\alpha+1}{2m+\alpha+\beta+2} \omega_m^{(\alpha+1,\beta)} P_m^{(\alpha+1,\beta)}(1) P_m^{(\alpha+1,\beta)}(\cos\theta) + \\
&\quad + d_{n+1} \cdot \frac{\alpha+1}{2n+\alpha+\beta+2} \omega_n^{(\alpha+1,\beta)} P_n^{(\alpha+1,\beta)}(1) P_n^{(\alpha+1,\beta)}(\cos\theta) = \sum_{s=m+1}^{n} I_s - J_1 + J_2,
\end{aligned} \tag{3.4}$$

say. Let N_θ be the smallest positive integer n such that $cn^{-1}\leqq\theta\leqq\pi-cn^{-1}$, where c is a positive constant in (2.6). By (2.1), (2.2) and (2.6), we get, for $n>m\geqq N_\theta$,

$$\sum_{s=m+1}^{n} |I_s| \leqq K_1 \theta^{-\alpha-3/2} (\pi-\theta)^{-\beta-1/2} \sum_{s=m+1}^{n} s^{\alpha+1/2} |\Delta d_s|,$$
$$|J_1| \leqq K_2 \theta^{-\alpha-3/2} (\pi-\theta)^{-\beta-1/2} m^{1/2} |a_m|,$$
$$|J_2| \leqq K_3 \theta^{-\alpha-3/2} (\pi-\theta)^{-\beta-1/2} n^{1/2} |a_n|.$$

Hence, from (3.1) and (3.4), we have, for $n>m\geqq N_\theta$,

$$\begin{aligned}
&\left| \sum_{s=m+1}^{n} a_s \omega_s^{(\alpha,\beta)} P_s^{(\alpha,\beta)}(\cos\theta) \right| \leqq \sum_{s=m+1}^{n} |I_s| + |J_1| + |J_2| \leqq \\
&\leqq K\theta^{-\alpha-3/2} (\pi-\theta)^{-\beta-1/2} \left(\sum_{s=m+1}^{n} s^{\alpha+1/2} |\Delta d_s| + m^{1/2}|a_m| + n^{1/2}|a_n| \right) \to 0 \quad \text{as} \quad n, m \to \infty.
\end{aligned} \tag{3.5}$$

Hence the Jacobi series (1.3) converges to a function $f(\theta)$ in $0<\theta<\pi$, and furthermore we get

$$f(\theta) = \sum_{n=0}^{\infty} (\Delta d_n) \frac{\alpha+1}{2n+\alpha+\beta+2} \omega_n^{(\alpha+1,\beta)} P_n^{(\alpha+1,\beta)}(1) P_n^{(\alpha+1,\beta)}(\cos\theta), \tag{3.6}$$

where the series converges absolutely in $0<\theta<\pi$ $\left(\omega_0^{(\alpha,\beta)} = (\alpha+1)\omega_0^{(\alpha+1,\beta)}/(\alpha+\beta+2)\right.$ and $\left. P_0^{(\alpha,\beta)}(x) = P_0^{(\alpha+1,\beta)}(x) = 1\right)$.

Next, let $\varepsilon\leqq\theta\leqq\pi-\varepsilon$ for every $0<\varepsilon<\pi$. We put $N_\theta = N_\varepsilon$ for $\theta=\varepsilon$. Then we can replace (3.5) by

$$\left| \sum_{s=m+1}^{n} a_s \omega_s^{(\alpha,\beta)} P_s^{(\alpha,\beta)}(\cos\theta) \right| \leqq K\varepsilon^{-\alpha-\beta-2} \left(\sum_{s=m+1}^{n} s^{\alpha+1/2} |\Delta d_s| + m^{1/2}|a_m| + n^{1/2}|a_n| \right)$$
$$\text{for} \quad n > m \geqq N_\varepsilon.$$

Hence (1.3) converges uniformly in $\varepsilon\leqq\theta\leqq\pi-\varepsilon$ to $f(\theta)$.

We prove (i). By $\alpha, \beta \geqq -1/2$ and $\mu, \nu > 0$, we get $\mu+\alpha+1/2>0$, $\mu+\alpha+1/2>$ $>\alpha-3/2$, $\nu+\beta-1/2>-1$ and $\nu+\beta-1/2>\beta-3/2$. Hence, from (3.6), (2.1), (2.2), (2.3), (2.9) and (3.1), we have

$$\int_0^\pi |f(\theta)|\theta^{\mu-\alpha-1/2}(\pi-\theta)^{\nu-\beta-3/2}\varrho^{(\alpha,\beta)}(\theta)\,d\theta \leqq$$

$$\leqq K_1 \int_0^\pi |f(\theta)| \left(\sin\frac{\theta}{2}\right)^{\mu+\alpha+1/2}\left(\cos\frac{\theta}{2}\right)^{\nu+\beta-1/2} d\theta \leqq$$

$$\leqq K_2 \sum_{n=0}^\infty |\Delta d_n| \frac{\alpha+1}{2n+\alpha+\beta+2}\,\omega_n^{(\alpha+1,\beta)} P_n^{(\alpha+1,\beta)}(1) \times$$

$$\times \int_0^\pi \left(\sin\frac{\theta}{2}\right)^{\mu+\alpha+1/2}\left(\cos\frac{\theta}{2}\right)^{\nu+\beta-1/2} |P_n^{(\alpha+1,\beta)}(\cos\theta)|\,d\theta \leqq$$

$$\leqq K_3|\Delta d_0| \int_{-1}^1 (1-x)^{(\mu+\alpha-1/2)/2}(1+x)^{(\nu+\beta-3/2)/2}\,dx +$$

$$+K_4 \sum_{n=1}^\infty n^{\alpha+1}|\Delta d_n| \int_{-1}^1 (1-x)^{(\mu+\alpha-1/2)/2}(1+x)^{(\nu+\beta-3/2)/2}|P_n^{(\alpha+1,\beta)}(x)|\,dx \leqq$$

$$\leqq K_5 + K_4 \sum_{n=1}^\infty n^{\alpha+1}|\Delta d_n| \left\{K_6 \int_0^1 (1-x)^{(\mu+\alpha-1/2)/2}|P_n^{(\alpha+1,\beta)}(x)|\,dx +\right.$$

$$\left.+K_7 \int_0^1 (1-x)^{(\nu+\beta-3/2)/2}|P_n^{(\beta,\alpha+1)}(x)|\,dx\right\} \leqq K_5 + K_7 \sum_{n=1}^\infty n^{\alpha+1/2}|\Delta d_n| < \infty.$$

Thus (i) is proved.

Finally, we prove (ii). From (3.6) we have, for any $0<\eta$, $\eta'<\pi/2$,

$$\int_\eta^{\pi-\eta'} f(\theta)\theta^{-\alpha-1/2}(\pi-\theta)^{-\beta-3/2}\varrho^{(\alpha,\beta)}(\theta)\,d\theta =$$

$$= \sum_{n=0}^\infty (\Delta b_n) \frac{\alpha+1}{2n+\alpha+\beta+2}\,\omega_n^{(\alpha+1,\beta)} P_n^{(\alpha+1,\beta)}(1) \times \tag{3.7}$$

$$\times \int_\eta^{\pi-\eta'} P_{n'}^{(\alpha+1,\beta)}(\cos\theta)\theta^{-\alpha-1/2}(\pi-\theta)^{-\beta-3/2}\varrho^{(\alpha,\beta)}(\theta)\,d\theta.$$

Now, by the second mean value theorem for integrals, we get

$$
\begin{aligned}
&\int_{\eta}^{\pi-\eta'} P_n^{(\alpha+1,\beta)}(\cos\theta)\,\theta^{-\alpha-1/2}(\pi-\theta)^{-\beta-3/2}\varrho^{(\alpha,\beta)}(\theta)\,d\theta = \\
&= 2^{-\alpha-\beta-2}\int_{\eta}^{\pi-\eta'} P_n^{(\alpha+1,\beta)}(\cos\theta)\left(\sin\frac{\theta}{2}\right)^{\alpha+1/2}\left(\cos\frac{\theta}{2}\right)^{\beta-1/2}\times \\
&\times\left(\frac{\sin\frac{\theta}{2}}{\frac{\theta}{2}}\right)^{\alpha+1/2}\left(\frac{\cos\frac{\theta}{2}}{\frac{\pi-\theta}{2}}\right)^{\beta+3/2} d\theta = 2^{-\alpha-\beta-2}\left(\frac{\sin\frac{\eta}{2}}{\frac{\eta}{2}}\right)^{\alpha+1/2}\left(\frac{\cos\frac{\pi-\eta'}{2}}{\frac{\eta'}{2}}\right)^{\beta+3/2}\times \\
&\times\int_{\psi}^{\pi-\psi'} P_n^{(\alpha+1,\beta)}(\cos\theta)\left(\sin\frac{\theta}{2}\right)^{\alpha+1/2}\left(\cos\frac{\theta}{2}\right)^{\beta-1/2} d\theta,
\end{aligned}
\tag{3.8}
$$

where we may assume $\eta \leqq \psi \leqq \pi/2$ and $\eta' \leqq \psi' \leqq \pi/2$ without loss of generality. We put

$$
\int_{\psi}^{\pi-\psi'} P_n^{(\alpha+1,\beta)}(\cos\theta)\left(\sin\frac{\theta}{2}\right)^{\alpha+1/2}\left(\cos\frac{\theta}{2}\right)^{\beta-1/2} d\theta = \int_{\psi}^{\pi/2} + \int_{\pi/2}^{\pi-\psi'} = Q_1 + Q_2,
\tag{3.9}
$$

say. First we estimate Q_1. We consider the case $\psi \leqq cn^{-1}$, where c is a positive constant in (2.6). We put

$$
\begin{aligned}
\sqrt{\omega_n^{(\alpha+1,\beta)}}\,Q_1 &= \left(\int_{\psi}^{c/n} + \int_{c/n}^{\pi/2}\right)\sqrt{\omega_n^{(\alpha+1,\beta)}}\,P_n^{(\alpha+1,\beta)}(\cos\theta)\left(\sin\frac{\theta}{2}\right)^{\alpha+1/2}\left(\cos\frac{\theta}{2}\right)^{\beta-1/2} d\theta = \\
&= Q_{1,1} + Q_{1,2}.
\end{aligned}
\tag{3.10}
$$

Then, from (2.1), (2.2) and (2.6), we have

$$
|Q_{1,1}| \leqq K\int_0^{c/n} n^{1/2} n^{\alpha+1}\theta^{\alpha+1/2}\,d\theta \leqq K_1 \qquad (\alpha \geqq -1/2).
\tag{3.11}
$$

Let $\lambda = (\alpha+\beta+2)/2$ and $\zeta = \alpha+3/2$. From (2.7), we get, with $\frac{c}{n} \leqq \sigma \leqq \frac{\pi}{2}$,

$$
\begin{aligned}
Q_{1,2} &= 2D\int_{c/n}^{\pi/2} \frac{\cos\left\{(n+\lambda)\theta - \frac{\pi\zeta}{2}\right\}}{\sin\theta}\,d\theta + O(n^{-1})\int_{c/n}^{\pi/2}(\sin\theta)^{-2}\,d\theta = \\
&= 2D\left(\sin\frac{c}{n}\right)^{-1}\int_{c/n}^{\sigma}\cos\left\{(n+\lambda)\theta - \frac{\pi\zeta}{2}\right\}d\theta + O(n^{-1})\int_{c/n}^{\pi/2}\theta^{-2}\,d\theta = O(1) \\
&\qquad \text{as} \quad n\to\infty.
\end{aligned}
\tag{3.12}
$$

Hence, from (3.10), (3.11) and (3.12), we have

$$
\sqrt{\omega_n^{(\alpha+1,\beta)}}\,|Q_1| \leqq K_2 \quad \text{for} \quad \psi \leqq cn^{-1}.
\tag{3.13}
$$

Moreover, by the same method as in (3.12) (replace cn^{-1} with ψ), we get

$$\sqrt{\omega_n^{(\alpha+1,\beta)}}|Q_1| \leqq K_3 \quad \text{for} \quad cn^{-1} \leqq \psi. \tag{3.14}$$

Thus (3.10). (3.13) and (3.14), we have

$$\sqrt{\omega_n^{(\alpha+1,\beta)}}|Q_1| \leqq K_4, \quad \text{where} \quad \eta \leqq \psi \leqq \pi/2. \tag{3.15}$$

Similarly we get

$$\sqrt{\omega_n^{(\alpha+1,\beta)}}|Q_2| \leqq K_5, \quad \text{where} \quad \eta' \leqq \psi' \leqq \pi/2. \tag{3.16}$$

On this occasion, we should notice in particular that, for $\psi' < cn^{-1}$,

$$\left| \int_{\pi-c/n}^{\pi-\psi'} \sqrt{\omega_n^{(\alpha+1,\beta)}} P_n^{(\alpha+1,\beta)}(\cos\theta)\left(\sin\frac{\theta}{2}\right)^{\alpha+1/2}\left(\cos\frac{\theta}{2}\right)^{\beta-1/2} d\theta \right| \leqq$$

$$\leqq K_6 \int_{\pi-c/n}^{\pi} n^{1/2} n^{\beta}(\pi-\theta)^{\beta-1/2}\, d\theta \leqq K_7 \qquad \left(\beta > -\frac{1}{2}\right)$$

by (2.1) and (2.6). Now, from (3.9), (3.15) and (3.16), we have

$$\sqrt{\omega_n^{(\alpha+1,\beta)}} \left| \int_{\psi}^{\pi-\psi'} P_n^{(\alpha+1,\beta)}(\cos\theta)\left(\sin\frac{\theta}{2}\right)^{\alpha+1/2}\left(\cos\frac{\theta}{2}\right)^{\beta-1/2} d\theta \right| \leqq K_8. \tag{3.17}$$

Thus, by (3.1), (2.1), (2.2), (3.8) and (3.17), we get

$$\sum_{n=0}^{\infty} \left| (\Delta b_n) \frac{\alpha+1}{2n+\alpha+\beta+2} \omega_n^{(\alpha+1,\beta)} P_n^{(\alpha+1,\beta)}(1) \times \right. \tag{3.18}$$

$$\left. \times \int_{\eta}^{\pi-\eta'} P_n^{(\alpha+1,\beta)}(\cos\theta)\theta^{-\alpha-1/2}(\pi-\theta)^{-\beta-3/2}\varrho^{(\alpha,\beta)}(\theta)\, d\theta \right| \leqq K_9 \sum_{n=0}^{\infty} n^{\alpha+1/2}|\Delta b_n| < \infty.$$

Hence, when we let $\eta, \eta' \to 0$, we have, from (3.7) and (3.18),

$$\int_{\to 0}^{\to \pi} f(\theta)\theta^{-\alpha-1/2}(\pi-\theta)^{-\beta-3/2}\varrho^{(\alpha,\beta)}(\theta)\, d\theta =$$

$$= \sum_{n=0}^{\infty} (\Delta b_n) \frac{\alpha+1}{2n+\alpha+\beta+2} \omega_n^{(\alpha+1,\beta)} P_n^{(\alpha+1,\beta)}(1) \times$$

$$\times \int_0^{\pi} P_n^{(\alpha+1,\beta)}(\cos\theta)\theta^{-\alpha-1/2}(\pi-\theta)^{-\beta-3/2}\varrho^{(\alpha,\beta)}(\theta)\, d\theta,$$

where the series converges absolutely. Thus (ii) is proved.

The proof of Theorem 1 is now complete.

For the proof of Theorem 2, we require the following lemma.

Lemma 2. *Let $\alpha \geqq -1/2$ and $\beta > -1/2$. Suppose that a sequence $\{a_n\}$ of real numbers satisfies the condition*

$$(3.19)\qquad \sum_{n=1}^{\infty} |\Delta \mathrm{d}_n| n^{\alpha+1/2} \log(n+1) < \infty \quad \textit{and} \quad n^{1/2} a_n \to 0 \quad \textit{as} \quad n \to \infty,$$

where $d_n = a_n / P_n^{(\alpha,\beta)}(1)$. Then condition (3.19) *is equivalent to conditions* (J3) *and* (J4).

Proof. If we assume (3.19), then we have for $\alpha \geqq -\frac{1}{2}$, by (2.1),

$$|a_n| n^{1/2} \log(n+1) = P_n^{(\alpha,\beta)}(1) n^{1/2} \log(n+1) \sum_{m=n}^{\infty} \Delta |\mathrm{d}_m| \leqq$$

$$\leqq K \sum_{m=n}^{\infty} |\Delta d_m| m^{\alpha+1/2} \log(m+1) \to 0 \quad \text{as} \quad n \to \infty.$$

Also, if we assume (J3) of (J4), then we get similarly

$$|a_n| n^{1/2} \log(u+1) \to 0 \quad \text{as} \quad n \to \infty.$$

Thus the proof of Lemma 2 is similar to that of Lemma 1.

Proof of Theorem 2. Conditions (J3) and (J4) are included in conditions (J1) and (J2), respectively. Hence the Jacobi series (1.3) converges to a function $f(\theta)$ in $0 < \theta < \pi$, and has the form (3.6). Now, from (2.1), (2.2), (2.9) and Lemma 2, we get for $\alpha \geqq -1/2$, $\beta \geqq -1/2$

$$\int_0^{\pi} |f(\theta)| \theta^{-\alpha-1/2} (\pi-\theta)^{-\beta-3/2} \varrho^{(\alpha,\beta)}(\theta)\, d\theta \leqq$$

$$\leqq \sum_{n=0}^{\infty} |\Delta d_n| \frac{\alpha+1}{2n+\alpha+\beta+2} \omega_n^{(\alpha+1,\beta)} P_n^{(\alpha+1,\beta)}(1) \times$$

$$\times \int_0^{\pi} |P_n^{(\alpha+1,\beta)}(\cos\theta)| \theta^{-\alpha-1/2} (\pi-\theta)^{-\beta-3/2} \varrho^{(\alpha,\beta)}(\theta)\, d\theta \leqq$$

$$\leqq K_1 |\Delta d_0| \int_{-1}^{1} (1-x)^{(\alpha-1/2)/2} (1+x)^{(\beta-3/2)/2}\, dx +$$

$$+ K_2 \sum_{n=1}^{\infty} |\Delta d_n| n^{\alpha+1} \int_{-1}^{1} (1-x)^{(\alpha-1/2)/2} (1+x)^{(\beta-3/2)/2} |P_n^{(\alpha+1,\beta)}(x)|\, dx \leqq$$

$$\leqq K_3 + K_4 \sum_{n=1}^{\infty} |\Delta d_n| n^{\alpha+1} \Big\{ K_5 \int_0^1 (1-x)^{(\alpha-1/2)/2} |P_n^{(\alpha+1,\beta)}(x)|\, dx +$$

$$+ K_6 \int_0^1 (1-x)^{(\beta-3/2)/2} |P_n^{(\beta,\alpha+1)}(x)|\, dx \Big\} \leqq K_3 + K_7 \sum_{n=1}^{\infty} |\Delta d_n| n^{\alpha+1/2} \log(n+1) < \infty.$$

Thus Theorem 2 is proved.

4. Proof of Theorem 3

For the proof of Theorem 3, we need the following lemma, due to S. ALJANČIĆ, R. BOJANIĆ and M. TOMIĆ [2].

Lemma 3. *Let $\mu>0$, and let $\varphi(u)$ be a slowly varying function. Suppose that $\{q_n\}$ is a non-increasing sequence tending to zero as $n\to\infty$. Then $\sum_{n=1}^{\infty} n^{\mu-1}\varphi(n)\, q_n$ converges if and only if $\sum_{n=1}^{\infty} n^{\mu}\varphi(u)\Delta q_n$ converges.*

Proof of Theorem 3. Since $\{n^{1/2}a_n\}$ $(n=1, 2, \dots)$ is non-increasing and tends to zero as $n\to\infty$, we have $\sum_{n=1}^{\infty}|\Delta(n^{1/2}a_n)|<\infty$. Moreover, since $\sum_{n=1}^{\infty} n^{-\delta+1/2}\varphi(n)|a_n|<<\infty$, we get $\sum_{n=1}^{\infty} n^{-1/2}|a_n|<\infty$ by (S2). Hence, from Remark 2, the sequence $\{a_n\}$ satisfies condition (J1). It is clear that $\{a_n\}$ satisfies condition (J2). Thus, by Theorem 1, the Jacobi series (1.3) converges to a function $f(\theta)$ in $0<\theta<\pi$, and has the form (3.6) (we put $d_n=a_n/P_n^{(\alpha,\beta)}(1)$).

First we shall prove (i). Hence we get, from (2.1) and (2.2),

$$\int_0^{\pi} |f(\theta)|\,\theta^{\delta-\alpha-3/2}(\pi-\theta)^{\nu-\beta-3/2}\varphi\left(\frac{1}{\theta(\pi-\theta)}\right)\varrho^{(\alpha,\beta)}(\theta)\,d\theta \leqq$$

$$\leqq \sum_{n=0}^{\infty}|\Delta d_n|\frac{\alpha+1}{2n+\alpha+\beta+2}\,\omega_n^{(\alpha+1,\beta)}P_n^{(\alpha+1,\beta)}(1)\int_0^{\pi}\theta^{\delta-\alpha-3/2}(\pi-\theta)^{\nu-\beta-3/2}\times$$

$$\times\varphi\left(\frac{1}{\theta(\pi-\theta)}\right)|P_n^{(\alpha+1,\beta)}(\cos\theta)|\,\varrho^{(\alpha,\beta)}(\theta)\,d\theta \leqq \tag{4.1}$$

$$\leqq K\sum_{n=0}^{\infty}|\Delta d_n|\,n^{\alpha+1}\Big(\int_0^{\pi/2}+\int_{\pi/2}^{\pi}\Big)\theta^{\delta+\alpha-1/2}(\pi-\theta)^{\nu+\beta-1/2}\varphi\left(\frac{1}{\theta(\pi-\theta)}\right)|P_n^{(\alpha+1,\beta)}(\cos\theta)|\,d\theta =$$

$$= K\sum_{n=0}^{\infty}|\Delta d_n|\,n^{\alpha+1}(U_n+V_n),$$

say. Since

$$\Delta d_n = \frac{\Delta(n^{1/2}a_n)}{n^{1/2}P_n^{(\alpha,\beta)}(1)}+\frac{(n+1)^{1/2}a_{n+1}}{n^{1/2}P_n^{(\alpha,\beta)}(1)}\left(1-\frac{n^{1/2}P_n^{(\alpha,\beta)}(1)}{(n+1)^{1/2}P_{n+1}^{(\alpha,\beta)}(1)}\right)$$

and by (1.1),

$$1-\frac{n^{1/2}P_n^{(\alpha,\beta)}(1)}{(n+1)^{1/2}P_{n+1}^{(\alpha,\beta)}(1)} = \frac{\alpha+\dfrac{1}{2}}{n}+O(n^{-2}),$$

and since $\{n^{1/2}a_n\}$ $(n=1, 2, \dots)$ is non-increasing, there is a positive integer N such that $\Delta d_n \geqq 0$ for all n, $n \geqq N$. Thus, from (4.1), we get

$$\int_0^\pi |f(\theta)|\theta^{\delta-\alpha-3/2}(\pi-\theta)^{\nu-\beta-3/2}\varphi\left(\frac{1}{\theta(\pi-\theta)}\right)\varrho^{(\alpha,\beta)}(\theta)\,d\theta \leqq \tag{4.2}$$
$$\leqq K_1 + K\sum_{n=N}^{\infty} n^{\alpha+1}(U_n+V_n)\Delta d_n.$$

By (S4), (2.6) and (S2), we have, for $n \geqq N$,

$$U_n \leqq K_2\left(n^{\alpha+1}\int_0^{c/n}\theta^{\delta+\alpha-1/2}\varphi\left(\frac{1}{\theta}\right)d\theta + n^{-1/2}\int_{c/n}^{\pi/2}\theta^{\delta+\alpha-1/2}\varphi\left(\frac{1}{\theta}\right)\cdot\theta^{-\alpha-3/2}\,d\theta\right) \leqq \tag{4.3}$$
$$\leqq K_3 n^{-\delta+1/2}\varphi\left(\frac{n}{c}\right) \leqq K_4 n^{-\delta+1/2}\varphi(n)$$

and

$$V_n \leqq K_4\left(\int_{\pi/2}^{\pi-c/n}(\pi-\theta)^{\nu+\beta-1/2}\varphi\left(\frac{1}{\pi-\theta}\right)n^{-1/2}(\pi-\theta)^{-\beta-1/2}d\theta + \right.$$
$$\left. + \int_{\pi-c/n}^{\pi}(\pi-\theta)^{\nu+\beta-1/2}\varphi\left(\frac{1}{\pi-\theta}\right)n^{\beta}\,d\theta\right) \leqq \tag{4.4}$$
$$\leqq \begin{cases} K_5\left(n^{-1/2}+n^{-\nu-1/2}\varphi(n)\right) & \text{for} \quad \nu > 0 \quad \text{and} \quad \frac{1}{2} > \beta \geqq -\frac{1}{2} \\ K_5 n^{-\nu-1/2}\varphi(n) & \text{for} \quad 0 > \nu > -\beta - \frac{1}{2} \quad \text{and} \quad \frac{1}{2} > \beta > -\frac{1}{2} \end{cases}$$
$$\leqq K n^{-\delta+1/2}\varphi(n) \qquad \text{for} \quad \nu \geqq \delta - 1.$$

Since $\sum_{n=1}^{\infty} n^{-\delta+1/2}\varphi(n)a_n$ converges, so does $\sum_{n=1}^{\infty} n^{-\delta+\alpha+1/2}\varphi(n)d_n$ by (2.1). Hence, if we put $q_n = d_n$ and $\mu = -\delta+\alpha+3/2$ in Lemma 3, then $\sum_{n=1}^{\infty} n^{-\delta+\alpha+3/2}\varphi(n)\Delta d_n$ converges. Now, by (4.2), (4.3) and (4.4), we have, for $\nu \neq 0$, $\nu \geqq \delta - 1$, $\nu > -\beta - 1/2$ and $1/2 > \beta \geqq -1/2$,

$$\int_0^\pi |f(\theta)|\theta^{\delta-\alpha-3/2}(\pi-\theta)^{\nu-\beta-3/2}\varphi\left(\frac{1}{\theta(\pi-\theta)}\right)\varrho^{(\alpha,\beta)}(\theta)\,d\theta \leqq$$
$$\leqq K + K_8\sum_{n=N}^{\infty} n^{-\delta+\alpha+3/2}\varphi(n)\Delta d_n < \infty.$$

Hence the case $\nu = 0$ and $1/2 > \beta > -1/2$ is clear. Thus (i) is proved.

Next, in order to prove (ii) it is enough to notice that for $\beta \geqq 1/2$

$$\int_{\pi/2}^{\pi} \theta^{\delta+\alpha-1/2}(\pi-\theta)^{\delta-\beta-5/2}\varphi\left(\frac{1}{\theta(\pi-\theta)}\right)|P_n^{(\alpha+1,\beta)}(\cos\theta)|\varrho^{(\alpha,\beta)}(\theta)\,d\theta \leqq$$

$$\leqq K\left\{\int_{\pi/2}^{\pi-c/n}(\pi-\theta)^{\delta-\beta-5/2}\varphi\left(\frac{1}{\pi-\theta}\right)n^{-1/2}(\pi-\theta)^{-\beta-1/2}(\pi-\theta)^{2\beta+1}d\theta+\right.$$

$$\left.+\int_{\pi-c/n}^{\pi}(\pi-\theta)^{\delta-\beta-5/2}\varphi\left(\frac{1}{\pi-\theta}\right)n^{\beta}(\pi-\theta)^{2\beta+1}d\theta\right\} \leqq K_1 n^{-\delta+1/2}\varphi(n)$$

by (S4), (2.6) and (S2). Thus Theorem 3 is proved.

5. Proofs of Theorem 4 and Corollary 1

For the proof of Theorem 4, we need the following three lemmas.

Lemma 4. *Let $\alpha, \beta \geqq -1/2$. Let $g(0) \in L([\theta, \pi]; \alpha, \beta)$ and let its Fourier—Jacobi series be*

$$g(\theta) \sim \sum_{n=0}^{\infty} b_n \omega_n^{(\alpha,\beta)} P_n^{(\alpha,\beta)}(\cos\theta).$$

Moreover, let

$$g_r(\theta) = \int_0^{\pi} g(v)\Psi_r(\theta, v)\varrho^{(\alpha,\beta)}(v)\,\mathrm{d}v = \sum_{n=0}^{\infty} b_n r^n \omega_n^{(\alpha,\beta)} P_n^{(\alpha,\beta)}(\cos\theta)$$

$$\text{for} \quad 0 \leqq \theta \leqq \pi \quad \text{and} \quad 0 \leqq r < 1,$$

where

$$\Psi_r(\theta, v) = \sum_{n=0}^{\infty} r^n \omega_n^{(\alpha,\beta)} P_n^{(\alpha,\beta)}(\cos\theta) P_n^{(\alpha,\beta)}(\cos v).$$

Then

$$\Psi_r(\theta, v) > 0 \quad \text{for} \quad 0 \leqq \theta,\ v \leqq \pi \quad \text{and} \quad 0 \leqq r < 1,$$

$$\int_0^{\pi} \Psi_r(\theta, v)\varrho^{(\alpha,\beta)}(v)\,dv = 1 \quad \text{for} \quad 0 \leqq \theta \leqq \pi \quad \text{and} \quad 0 \leqq r < 1,$$

and $g_r(\theta)$ converges to $g(\theta)$ as $r \to 1-0$ for almost every $0 \leqq \theta \leqq \pi$.

Lemma 4 is due to H. Bavinck [5; p. 4].

Lemma 5. *Let $\alpha, \beta \geqq -1/2$ and let $f(\theta) \in L([0, \pi]; \alpha, \beta)$. Suppose that $g(\theta)$ is a bounded and Lebesgue-measurable function on* [0, π]. *Then $f(\theta)g(\theta) \in L([0, \pi]; \alpha, \beta)$ and the formula*

$$\int_0^{\pi} f(\theta)g(\theta)\varrho^{(\alpha,\beta)}(\theta)\,\mathrm{d}\theta = \lim_{r \to 1-0} \sum_{n=0}^{\infty} a_n b_n \omega_n^{(\alpha,\beta)} r^n$$

holds, where a_n, b_n denote the Fourier—Jacobi coefficients of the functions f, g respectively.

Proof. We have clearly $g(\theta)\in L([0,\pi];\alpha,\beta)$ and $f(\theta)g(\theta)\in L([0,\pi];\alpha,\beta)$. We define $g_r(\theta)$ and $\Psi_r(\theta,v)$ as in Lemma 4. Since $g(\theta)$ is bounded and Lebesgue-measurable, we have, by Lemma 4,

$$|g_r(\theta)|\leqq\int_0^\pi |g(v)|\Psi_r(\theta,v)\varrho^{(\alpha,\beta)}(v)\,dv\leqq\sup_{0\leqq v\leqq\pi}|g(v)|\quad\text{for}\quad 0\leqq\theta\leqq\pi\quad\text{and}\quad 0\leqq r<1,$$

and further $g_r(\theta)$ converges to $g(\theta)$ as $r\to 1-0$ for almost every $0\leqq\theta\leqq\pi$. Hence

$$\int_0^\pi f(\theta)g(\theta)\varrho^{(\alpha,\beta)}(\theta)\,d\theta=\lim_{r\to1-0}\int_0^\pi f(\theta)g_r(\theta)\varrho^{(\alpha,\beta)}(\theta)\,d\theta=\lim_{r\to1-0}\sum_{n=0}^\infty a_nb_n\omega_n^{(\alpha,\beta)}r^n.$$

Thus Lemma 5 is proved.

Lemma 6. *If $H(\theta)$ is a non-negative and non-increasing function on* $(0,\infty)$ *and is Lebesgue-integrable in any finite interval, then*

$$\Big|\int_0^v H(\theta)\cos\theta\,d\theta\Big|\leqq\int_0^{\pi/2}H(\theta)\,d\theta\quad\text{and}\quad 0\leqq\int_0^v H(\theta)\sin\theta\,d\theta\leqq\int_0^\pi H(\theta)\,d\theta$$

for any $v>0$.

Lemma 6 is due to M. and S. Izumi [8; Lemma 1 and 2].

Proof of Theorem 4. By assumption, it is clear that $g(\theta)$ is bounded and Lebesgue-integrable on $[\eta',\eta'']$ for any $0<\eta'<\eta''<\pi$. Hence, since $f(\theta)\in L([0,\pi];\alpha,\beta)$, we have, by Lemma 5,

$$\int_{\eta'}^{\eta''} f(\theta)g(\theta)\varrho^{(\alpha,\beta)}(\theta)\,d\theta=\lim_{r\to1-0}\sum_{n=0}^\infty a_n\omega_n^{(\alpha,\beta)}r^n\int_{\eta'}^{\eta''}g(\theta)P_n^{(\alpha,\beta)}(\cos\theta)\varrho^{(\alpha,\beta)}(\theta)\,d\theta.$$

First we shall prove that $\int_{\eta'}^{\eta''} f(\theta)g(\theta)\varrho^{(\alpha,\beta)}(\theta)\,d\theta$ converges to zero as η' and η'' tend to zero, i.e., that $\int_{\to0}^{\pi/2} f(\theta)g(\theta)\varrho^{(\alpha,\beta)}(\theta)\,d\theta$ exists and is finite. Let $0<\eta'<\eta''\leqq\leqq\pi/2$. We have, for any positive integer N,

$$\int_{\eta'}^{\eta''} f(\theta)g(\theta)\varrho^{(\alpha,\beta)}(\theta)\,d\theta=$$

$$=\lim_{r\to1-0}\left\{\sum_{n=0}^N a_n\omega_n^{(\alpha,\beta)}r^n\int_{\eta'}^{\eta''}g(\theta)P_n^{(\alpha,\beta)}(\cos\theta)\varrho^{(\alpha,\beta)}(\theta)\,d\theta+\sum_{n=N+1}^\infty a_n\omega_n^{(\alpha,\beta)}r^n\big(I(\eta'')-I(\eta')\big)\right\},\tag{5.1}$$

where

$$I(\eta)=\int_0^\eta g(\theta)P_n^{(\alpha,\beta)}(\cos\theta)\varrho^{(\alpha,\beta)}(\theta)\,d\theta\quad\text{for}\quad 0<\eta\leqq\pi/2.$$

We consider the case $\eta<c/n\leqq\pi/2$, where c is the positive constant in (2.6). Since $\theta^{\alpha+1/2}g(\theta)$ is non-negative, Lebesgue-integrable and non-increasing on $(0, \pi/2)$, we have, by (2.6),

$$|I(\eta)| \leqq K\int_0^{1/n} g(\theta)n^\alpha\left(\sin\frac{\theta}{2}\right)^{2\alpha+1}d\theta \leqq K_1 n^{-1/2}\int_0^{1/n} g(\theta)\theta^{\alpha+1/2}\,d\theta \quad \text{for} \quad \eta<\frac{c}{n}\leqq\frac{\pi}{2}. \tag{5.2}$$

Next we consider the case $c/n\leqq\eta$. We set $\lambda=(\alpha+\beta+1)/2$ and $\zeta=\alpha+1/2$. By (2.8), we have

$$\sqrt{\omega_n^{(\alpha,\beta)}}I(\eta) = (D+D^*n^{-1})T_1(\eta)+T_2(\eta)+O(n^{-2})T_3(\eta), \tag{5.3}$$

where

$$T_1(\eta) = \int_{c/n}^{\eta} g(\theta)\left(\sin\frac{\theta}{2}\right)^{\alpha+1/2}\left(\cos\frac{\theta}{2}\right)^{\beta+1/2}\cos\left\{(n+\lambda)\theta-\frac{\pi\zeta}{2}\right\}d\theta,$$

$$T_2(\eta) = \int_{c/n}^{\eta} g(\theta)\left(\sin\frac{\theta}{2}\right)^{\alpha+1/2}\left(\cos\frac{\theta}{2}\right)^{\beta+1/2}\left(En^{-1}\cot\frac{\theta}{2}+E^*n^{-1}\tan\frac{\theta}{2}\right)\times$$

$$\times\sin\left\{(n+\lambda)\theta-\frac{\pi\zeta}{2}\right\}d\theta,$$

$$T_3(\eta) = \int_{c/n}^{\eta} g(\theta)\left(\sin\frac{\theta}{2}\right)^{\alpha-3/2}\left(\cos\frac{\theta}{2}\right)^{\beta-3/2}d\theta.$$

Since $\theta^{\alpha+1/2}g(\theta)$ is non-negative and non-increasing on $(0, \pi/2)$, so is $(\sin\theta/2)^{\alpha+1/2}g(\theta)$. Now, without loss of generality, we may assume that $\cos\lambda\theta$ is positive and decreasing on $[0, \eta]$. Then, from Lemma 6,

$$\left|\int_{c/n}^{\eta} g(\theta)\left(\sin\frac{\theta}{2}\right)^{\alpha+1/2}\left(\cos\frac{\theta}{2}\right)^{\beta+1/2}\cos\lambda\theta\cos n\theta\,d\theta\right| \leqq$$

$$\leqq K_2\int_0^{1/n} g(\theta)\left(\sin\frac{\theta}{2}\right)^{\alpha+1/2}\left(\cos\frac{\theta}{2}\right)^{\beta+1/2}\cos\lambda\theta\,d\theta \leqq K_3\int_0^{1/n}\theta^{\alpha+1/2}g(\theta)\,d\theta.$$

Similary we have, by the second mean value theorem for integrals and Lemma 6,

$$|T_1(\eta)| \leqq K_4\int_0^{1/n}\theta^{\alpha+1/2}g(\theta)\,d\theta \quad \text{for} \quad \frac{c}{n}\leqq\eta. \tag{5.4}$$

Similarly we get, for $cn^{-1} \leqq \eta$, with $\frac{c}{n} \leqq \psi$, $\psi' \leqq \eta$,

$$|T_2(\eta)| = \left|\left(\frac{E}{n}\cot\frac{c}{2n}\right)\int_{c/n}^{\psi} g(\theta)\left(\sin\frac{\theta}{2}\right)^{\alpha+1/2}\left(\cos\frac{\theta}{2}\right)^{\beta+1/2}\sin\left\{(n+\lambda)\theta-\frac{\pi\zeta}{2}\right\}d\theta + \right.$$

$$\text{(5.5)} \quad \left. +\left(\frac{E^*}{n}\tan\frac{\eta}{2}\right)\int_{\psi'}^{\eta} g(\theta)\left(\sin\frac{\theta}{2}\right)^{\alpha+1/2}\left(\cos\frac{\theta}{2}\right)^{\beta+1/2}\sin\left\{(n+\lambda)\theta-\frac{\pi\zeta}{2}\right\}d\theta\right| \leqq$$

$$\leqq K_5 \int_0^{1/n} \theta^{\alpha+1/2} g(\theta)\,d\theta.$$

Hence, by (5.3), (5.4), (5.5) and (2.2), we obtain

$$\text{(5.6)} \qquad |I(\eta)| \leqq K_6 n^{-1/2}\int_0^{1/n} \theta^{\alpha+1/2} g(\theta)\,d\theta \quad \text{for} \quad \frac{c}{n} \leqq \eta.$$

Thus, by (5.2) and (5.6),

$$|I(\eta)| \leqq K_7 n^{-1/2}\int_0^{1/n} \theta^{\alpha+1/2} g(\theta)\,d\theta \quad \text{for} \quad 0 < \eta \leqq \frac{\pi}{2} \quad \text{and all} \quad n \geqq \left[\frac{2c}{\pi}\right]+1.$$

In this estimation, we can replace η by η' or η'', $0<\eta'<\eta''\leqq\pi/2$. Hence, by (5.1), (2.2) and (2.6), we have

$$\text{(5.7)} \qquad \left|\int_{\eta'}^{\eta''} f(\theta)g(\theta)\varrho^{(\alpha,\beta)}(\theta)\,d\theta\right| \leqq$$
$$\leqq K_8 \sum_{n=1}^{N}|a_n|(n+1)^{\alpha+1}\int_{\eta}^{\eta'} g(\theta)\varrho^{(\alpha,\beta)}(\theta)\,d\theta + K_9 \sum_{n=N+1}^{\infty} n^{1/2}|a_n|\int_0^{1/n}\theta^{\alpha+1/2}g(\theta)\,d\theta.$$

We have $\sum_{n=1}^{\infty} n^{1/2}|a_n| \int_0^{1/n} \theta^{\alpha+1/2} g(\theta)\, d\theta<\infty$ by assumption, so for any $\varepsilon>0$ we can take N so that the second term on the right-hand side is less than $\varepsilon/2$, and then take η' and η so near to zero that the first term on the right-hand side is less than $\varepsilon/2$. Then we have

$$\left|\int_{\eta'}^{\eta''} f(\theta)g(\theta)\varrho^{(\alpha,\beta)}(\theta)\,d\theta\right| < \varepsilon$$

for all η' and η'' sufficiently near zero. Thus,

$$\text{(5.8)} \qquad \lim_{\eta\to+0}\int_{\eta}^{\pi/2} f(\theta)g(\theta)\varrho^{(\alpha,\beta)}(\theta)\,d\theta = \int_{\to 0}^{\pi/2} f(\theta)g(\theta)\varrho^{(\alpha,\beta)}(\theta)\,d\theta$$

exists and is finite. Furthermore we have, by (5.1) and (5.7),

$$\int_{\to 0}^{\pi/2} f(\theta)g(\theta)\varrho^{(\alpha,\beta)}(\theta)\,d\theta =$$

$$(5.9)\quad \begin{aligned} &= \lim_{\eta\to+0}\lim_{r\to 1-0}\sum_{n=0}^{\infty} a_n\omega_n^{(\alpha,\beta)} r^n \int_{\eta}^{\pi/2} g(\theta)P_n^{(\alpha,\beta)}(\cos\theta)\varrho^{(\alpha,\beta)}(\theta)\,d\theta = \\ &= \lim_{\eta\to+0}\sum_{n=0}^{\infty} a_n\omega_n^{(\alpha,\beta)} \int_{\eta}^{\pi/2} g(\theta)P_n^{(\alpha,\beta)}(\cos\theta)\varrho^{\alpha,\beta)}(\theta)\,d\theta = \\ &= \sum_{n=0}^{\infty} a_n\omega_n^{(\alpha,\beta)} \int_{0}^{\pi/2} g(\theta)P_n^{(\alpha,\beta)}(\cos\theta)\varrho^{(\alpha,\beta)}(\theta)\,d\theta, \end{aligned}$$

where the last series converges absolutely. Similarly we can prove that

$$(5.10)\qquad \lim_{\eta\to\pi-0}\int_{\pi/2}^{\eta} f(\theta)g(\theta)\varrho^{(\alpha,\beta)}(\theta)\,d\theta = \int_{\pi/2}^{\to\pi} f(\theta)g(\theta)\varrho^{(\alpha,\beta)}(\theta)\,d\theta$$

exists and is finite, and that

$$(5.11)\quad \int_{\pi/2}^{\to\pi} f(\theta)g(\theta)\varrho^{(\alpha,\beta)}(\theta)\,d\theta = \sum_{n=0}^{\infty} a_n\omega_n^{(\alpha,\beta)} \int_{\pi/2}^{\pi} g(\theta)P_n^{(\alpha,\beta)}(\cos\theta)\varrho^{(\alpha,\beta)}(\theta)\,d\theta,$$

where the series converges absolutely. Hence, from (5.8), (5.9), (5.10) and (5.11), $f(\theta)g(\theta)\in R([0,\pi];\alpha,\beta)$ and Parseval's formula

$$\int_{\to 0}^{\to\pi} f(\theta)g(\theta)\varrho^{(\alpha,\beta)}(\theta)\,d\theta = \sum_{n=0}^{\infty} a_n\omega_n^{(\alpha,\beta)} \int_{0}^{\pi} g(\theta)P_n^{(\alpha,\beta)}(\cos\theta)\varrho^{(\alpha,\beta)}(\theta)\,d\theta = \sum_{n=0}^{\infty} a_n b_n\omega_n^{(\alpha,\beta)}$$

holds, where the last series converges absolutely. Thus Theorem 4 is proved.

Proof of Corollary 1. We notice that the function $\theta(\pi-\theta)$ is increasing on $0\leqq\theta\leqq\pi/2$ and decreasing on $\pi/2\leqq\theta\leqq\pi$. If we put

$$g(\theta) = \theta^{\delta-\alpha-3/2}\varphi\left(\frac{1}{\theta(\pi-\theta)}\right) \quad \text{for} \quad 0<\theta\leqq\frac{\pi}{2},$$

and

$$g(\theta) = (\pi-\theta)^{\delta-\beta-3/2}\varphi\left(\frac{1}{\theta(\pi-\theta)}\right) \quad \text{for} \quad \frac{\pi}{2}<\theta<\pi$$

in Theorem 4, then

$$\sum_{n=1}^{\infty} n^{1/2}|a_n|\Big(\int_0^{1/n}\theta^{\alpha+1/2}g(\theta)\,d\theta + \int_0^{1/n}\theta^{\beta+1/2}g(\pi-\theta)\,d\theta\Big) \leqq K\sum_{n=1}^{\infty} n^{-\delta+1/2}\varphi(n)|a_n| < \infty.$$

Hence we have $f(\theta)g(\theta)\in R([0,\pi];\alpha,\beta)$ by Theorem 4. Now, for any $\varepsilon>0$, there is a positive number $\Theta\leqq\pi/2$ such that

$$\left|\int_{\eta}^{\eta'} f(\theta)g(\theta)\varrho^{(\alpha,\beta)}(\theta)\,d\theta\right|<\varepsilon \quad \text{for all } \eta \text{ and } \eta', \quad 0<\eta<\eta'<\Theta.$$

Thus, by the second mean value theorem for integrals, we have, with $\eta\leqq\Psi\leqq\eta'$,

$$\left|\int_{\eta}^{\eta'} f(\theta)\,\theta^{\delta-\alpha-3/2}(\pi-\theta)^{\delta-\beta-3/2}\varphi\left(\frac{1}{\theta(\pi-\theta)}\right)\varrho^{\alpha,\beta}(\theta)\,d\theta\right|=$$

$$=(\pi-\eta')^{\delta-\beta-3/2}\left|\int_{\Psi}^{\eta'} f(\theta)g(\theta)\varrho^{(\alpha,\beta)}(\theta)\,d\theta\right|\leqq\left(\frac{\pi}{2}\right)^{\delta-\beta-3/2}\varepsilon.$$

Hence

$$\int_{\to 0}^{\pi/2} f(\theta)\,\theta^{\delta-\alpha-3/2}(\pi-\theta)^{\delta-\beta-3/2}\varphi\left(\frac{1}{\theta(\pi-\theta)}\right)\varrho^{(\alpha,\beta)}(\theta)\,d\theta$$

exists and is finite. Similarly,

$$\int_{\pi/2}^{\to\pi} f(\theta)\,\theta^{\delta-\alpha-3/2}(\pi-\theta)^{\delta-\beta-3/2}\varphi\left(\frac{1}{\theta(\pi-\theta)}\right)\varrho^{(\alpha,\beta)}(\theta)\,d\theta$$

exists and is finite. Thus,

$$f(\theta)\,\theta^{\delta-\alpha-3/2}(\pi-\theta)^{\delta-\beta-3/2}\varphi\left(\frac{1}{\theta(\pi-\theta)}\right)\in R([0,\pi];\alpha,\beta)$$

and Corollary 1 is proved.

6. Proofs of Theorem 5 and Corollaries 2, 3

For the proof of Theorem 5, we require the following two lemmas.

Lemma 7. *If $\{q_n\}$ is a non-negative and non-increasing sequence, then, for $0\leqq m\leqq N\leqq\infty$, $0\leqq\theta\leqq\pi$ and any real numbers u, v, we have*

$$\left|\sum_{n=m}^{N} q_n e^{i\{(n+u)\theta+v\}}\right|\leqq\sum_{n=0}^{[\theta^{-1}]} q_n \quad \textit{for any } m, \textit{ and } \quad \leqq K\theta^{-1}q_m \quad \textit{for} \quad m\geqq\left[\frac{K_1}{\theta}\right].$$

Lemma 7 is due to L. MacFadden [10].

Lemma 8. Let $\alpha,\beta\geqq-1/2$. *Suppose that the Jacobi series* (1.5) *satisfies conditions* (J5), (J6) *and* (J7). *Define $G(\theta)$ as in* (1.6). *Then the series* (1.5) *converges* *to a function* $g(\theta)$ *in* $0<\theta<\pi$, *and the following four inequalities hold:*

(6.1) $$\left|\sum_{n=0}^{m} b_n\omega_n^{(\alpha,\beta)}P_n^{(\alpha,\beta)}(\cos\theta)\right| \leqq K_1G(\theta) \quad \textit{for any } m \textit{ and } \theta,\ 0\leqq\theta\leqq\pi,$$

(6.2) $$\left|\sum_{n=m}^{N} b_n\omega_n^{(\alpha,\beta)}P_n^{(\alpha,\beta)}(\cos\theta)\right| \leqq K_2\theta^{-\alpha-3/2}m^{1/2}b_m \leqq K_3g(\theta)$$

$$\textit{for any}\quad m,\ N,\ \ N\geqq m\geqq\left[\frac{c}{\theta}\right]+1,\quad 0<\theta\leqq\frac{\pi}{2},$$

(6.3) $$\left|\sum_{n=m}^{N} b_n\omega_n^{(\alpha,\beta)}P_n^{(\alpha,\beta)}(\cos\theta)\right| \leqq K_4(\pi-\theta)^{-\beta-1/2}m^{1/2}b_m \leqq K_5G(\theta)$$

$$\textit{for any}\quad m,\ N,\ N\geqq m\geqq\left[\frac{c}{\pi-\theta}\right]+1,\quad \frac{\pi}{2}<\theta<\pi,$$

(6.4) $$|g(\theta)|\leqq G(\theta)\quad \textit{for}\quad 0\leqq\theta\leqq\pi,$$

where c *is a positive constant in* (2.6).

Proof. By (J5) and (J6), we notice $b_n\geqq 0$ for $n=0, 1, 2, \ldots$. Further, from (J6), we have

$$\sum_{n=1}^{\infty}|\Delta(n^{1/2}b_n)|<\infty \quad\text{and}\quad n^{1/2}b_n\to 0 \quad\text{as}\quad n\to\infty.$$

Hence, by (J7) and Remark 2, the Jacobi series (1.5) satisfies (J1). Also, (1.5) satisfies clearly (J2). Thus, by Theorem 1, (1.5) converges to a function $g(\theta)$ in $0<\theta<\pi$.

First, we shall prove (6.2) and the case $0\leqq\theta\leqq\pi/2$ of (6.1). We put $0<\theta\leqq\pi/2$. By (2.2), (2.6) and (J6), we have, for $0\leqq m\leqq[c/\theta]$,

(6.5) $$\left|\sum_{n=0}^{m} b_n\omega_n^{(\alpha,\beta)}P_n^{(\alpha,\beta)}(\cos\theta)\right| \leqq K\sum_{n=0}^{[c/\theta]} b_n(n+1)^{\alpha+1}\leqq K_1G(\theta).$$

By Abel's transformation, we get, for $N\geqq m\geqq[c/\theta]+1$,

$$\sum_{n=m}^{N} b_n\omega_n^{(\alpha,\beta)}P_n^{(\alpha,\beta)}(\cos\theta) = \sum_{n=m}^{N}(\Delta b_n^*)\frac{\alpha+1}{2n+\alpha+\beta+2}\omega_n^{(\alpha+1,\beta)}P_n^{(\alpha+1,\beta)}(1)P_n^{(\alpha+1,\beta)}(\cos\theta)-$$

(6.6) $$-b_m^*\cdot\frac{\alpha+1}{2m+\alpha+\beta}\omega_{m-1}^{(\alpha+1,\beta)}P_{m-1}^{(\alpha+1,\beta)}(1)P_{m-1}^{(\alpha+1,\beta)}(\cos\theta)+$$

$$+b_{N+1}^*\cdot\frac{\alpha+1}{2N+\alpha+\beta+2}\omega_N^{(\alpha+1,\beta)}P_N^{(\alpha+1,\beta)}(1)P_N^{(\alpha+1,\beta)}(\cos\theta)=X_1-X_2+X_3,$$

say, where $b_n^* = b_n/P_n^{(\alpha,\beta)}(1)$. We have, for $N \geqq m \geqq \left[\frac{c}{\theta}\right]+1$,

$$X_1 = \sum_{n=m}^{N} \frac{\Delta(n^{1/2}b_n)}{n^{1/2}P_n^{(\alpha,\beta)}(1)} \cdot \frac{\alpha+1}{2n+\alpha+\beta+2}\,\omega_n^{(\alpha+1,\beta)}P_n^{(\alpha+1,\beta)}(1)\,P_n^{(\alpha+1,\beta)}(\cos\theta)+ \tag{6.7}$$

$$+\sum_{n=m}^{N}(n+1)^{1/2}b_{n+1}\left(\Delta\frac{1}{n^{1/2}P_n^{(\alpha,\beta)}(1)}\right)\frac{\alpha+1}{2n+\alpha+\beta+2}\,\omega_n^{(\alpha+1,\beta)}\cdot P_n^{(\alpha+1,\beta)}(1)P_n^{(\alpha+1,\beta)}(\cos\theta)=$$

$$= X_{1,1}+X_{1,2},$$

say. By (J6), (2.1), (2.2) and (2.6), we get

$$|X_{1,1}| \leqq K_2\theta^{-\alpha-3/2}\sum_{n=m}^{N}\Delta(n^{1/2}b_n) \leqq K_2\theta^{-\alpha-3/2}m^{1/2}b_m. \tag{6.8}$$

By (1.1), (2.2), (2.7), (J6) and Lemma 7 (put $K_1=c$), we have

$$X_{1,2} = \sum_{n=m}^{N}(n+1)^{1/2}b_{n+1}\left\{\frac{F}{n}+O\left(\frac{1}{n^2}\right)\right\}\left(\sin\frac{\theta}{2}\right)^{-\alpha-3/2}\left(\cos\frac{\theta}{2}\right)^{-\beta-1/2}\times$$

$$\times\left(A\cdot\cos\left\{\left(n+\frac{\alpha+\beta+2}{2}\right)\theta-\frac{\pi}{2}\left(\alpha+\frac{3}{2}\right)\right\}+(\sin\theta)^{-1}O(n^{-1})\right)=$$

$$= O(1)\theta^{-\alpha-3/2}\sum_{n=m}^{N}(n+1)^{-1/2}b_{n+1}\cos\left\{\left(n+\frac{\alpha+\beta+2}{2}\right)\theta-\frac{\pi}{2}\left(\alpha+\frac{3}{2}\right)\right\}+$$

$$+O(1)\theta^{-\alpha-3/2}\sum_{n=m}^{N}(n+1)^{-3/2}b_{n+1}+O(1)\theta^{-\alpha-5/2}\sum_{n=m}^{N}(n+1)^{-3/2}b_{n+1}+ \tag{6.9}$$

$$+O(1)\theta^{-\alpha-5/2}\sum_{n=m}^{N}(n+1)^{-5/2}b_{n+1}=$$

$$= O(1)\theta^{-\alpha-3/2}\theta^{-1}(m+1)^{-1/2}b_{m+1}+O(1)\theta^{-\alpha-3/2}(m+1)^{-1/2}b_{m+1}+$$

$$+O(1)\theta^{-\alpha-5/2}(m+1)^{-1/2}b_{m+1}+O(1)\theta^{-\alpha-5/2}(m+1)^{-3/2}b_{m+1}=O(1)\theta^{-\alpha-3/2}m^{1/2}b_m$$

$$\text{as}\quad m\to\infty,$$

where F is a constant depending only on α and β. By (6.7), (6.8) and (6.9), we have

$$|X_1| \leqq K_3\theta^{-\alpha-3/2}m^{1/2}b_m. \tag{6.10}$$

Next, from (6.6), (2.1), (2.2), (2.6) and (J6), we get

$$|X_2| \leqq K_4 b_m^* m^{\alpha+1}m^{-1/2}\theta^{-\alpha-3/2} \leqq K_5\theta^{-\alpha-3/2}m^{1/2}b_m \tag{6.11}$$

and

$$|X_3| \leqq K_6\theta^{-\alpha-3/2}N^{1/2}b_{N+1} \leqq K_7\theta^{-\alpha-3/2}m^{1/2}b_m. \tag{6.12}$$

Hence, from (6.6), (6.10), (6.11), (6.12) and (J6), we have, for $N \geqq m \geqq [c/\theta]+1$,

$$\left|\sum_{n=m}^{N} b_n \omega_n^{(\alpha,\beta)} P_n^{(\alpha,\beta)}(\cos\theta)\right| \leqq K_8 \theta^{-\alpha-3/2} m^{1/2} b_m \leqq K_9 \theta^{-\alpha-2} b_{[c/\theta]+1} \leqq \tag{6.13}$$
$$\leqq K \sum_{n=0}^{[c/\theta]} (n+1)^{\alpha+1} b_n \leqq K_1 G(\theta).$$

Thus we get (6.2). By (6.5) and (6.13), we have the case $0<\theta\leqq\pi/2$ of (6.1). The case $\theta=0$ of (6.1) is trivial from (2.1) and (2.2). Hence we get the case $0\leqq\theta\leqq\pi/2$ of (6.1).

Secondly, we shall prove (6.3) and the case $\pi/2<\theta\leqq\pi$ of (6.1). We put $-1\leqq x<0$. By (1.2), (2.3) and Abel's transformation, we have, for $m=0, 1, 2, \dots$,

$$\sum_{n=0}^{m} b_n \omega_n^{(\alpha,\beta)} P_n^{(\alpha,\beta)}(x) = b_0 \omega_0^{(\beta,\alpha)} + \sum_{n=1}^{m} (-1)^n b_n \omega_n^{(\beta,\alpha)} P_n^{(\beta,\alpha)}(-x) =$$
$$= b_0 \omega_0^{(\beta,\alpha)} + \sum_{n=1}^{m} \left\{\sum_{s=1}^{n} (-1)^s\right\} \Delta \{b_n \omega_n^{(\beta,\alpha)} P_n^{(\beta,\alpha)}(-x)\} + \tag{6.14}$$
$$+ \left\{\sum_{s=1}^{m} (-1)^s\right\} b_{m+1} \omega_{m+1}^{(\beta,\alpha)} P_{m+1}^{(\beta,\alpha)}(-x) = b_0 \omega_0^{(\beta,\alpha)} + Y_1 + Y_2,$$

say. From (2.2) and (2.5), we get, for $n=1, 2, \dots$,

$$\Delta\left(b_n \omega_n^{(\beta,\alpha)} P_n^{(\beta,\alpha)}(-x)\right) =$$
$$= 2\left(b_n n\left\{1+O\left(\frac{1}{n}\right)\right\} P_n^{(\beta,\alpha)}(-x) - b_{n+1}(n+1)\left\{1+O\left(\frac{1}{n}\right)\right\} P_{n+1}^{(\beta,\alpha)}(-x)\right) =$$
$$= 2\left(b_n\{(n+\beta+1) P_n^{(\beta,\alpha)}(-x) - (n+1) P_{n+1}^{(\beta,\alpha)}(-x)\} + O(1) b_n P_n^{(\beta,\alpha)}(-x) + \right. \tag{6.15}$$
$$\left. + O(1) b_{n+1} P_{n+1}^{(\beta,\alpha)}(-x) - (\beta+1) b_n P_n^{(\beta,\alpha)}(-x) + (\Delta b_n)(n+1) P_{n+1}^{(\beta,\alpha)}(-x)\right) =$$
$$= 2\left\{b_n\left(n+\frac{\alpha+\beta+2}{2}\right)(1+x) P_n^{(\beta+1,\alpha)}(-x) + O(1) b_n P_n^{(\beta,\alpha)}(-x) + \right.$$
$$\left. + O(1) b_{n+1} P_{n+1}^{(\beta,\alpha)}(-x) + (\Delta b_n)(n+1) P_{n+1}^{(\beta,\alpha)}(-x)\right\}.$$

By Abel's transformation, we have, for $m=1, 2, \dots$,

$$\sum_{n=1}^{m} n^{\beta+1} \Delta b_n = \sum_{n-1}^{m-1} (-b_{n+1}) \Delta n^{\beta+1} + b_1 - b_{m+1} m^{\beta+1} \leqq K \sum_{n=1}^{m} n^\beta b_n. \tag{6.16}$$

By (J6), the sequence $\{b_n\}$ is non-increasing. From (6.14), (6.15), (2.1) and (6.16), we get, for $m=1, 2, \ldots$,

$$\begin{aligned} |Y_1| &\leqq K\left\{(1+x)\sum_{n=1}^{m} n^{\beta+2}b_n + \sum_{n=1}^{m} n^{\beta}b_n + \sum_{n=1}^{m} n^{\beta+1}\Delta b_n\right\} \leqq \\ &\leqq K_1\left\{(1+x)\sum_{n=0}^{m}(n+1)^{\beta+2}b_n + \sum_{n=0}^{m}(n+1)^{\beta}b_n\right\}. \end{aligned} \tag{6.17}$$

By (6.14), (2.1), (2.2) and (J6), we get, for $m=1, 2, \ldots$,

$$|Y_2| \leqq K(m+1)^{\beta+1}b_{m+1} \leqq K(m+1)^{\beta+1}b_m \leqq K_1\sum_{n=0}^{m}(n+1)^{\beta}b_n. \tag{6.18}$$

Thus, from (6.14), (6.17) and (6.18), we have, for $-1\leqq x<0$ and $m=0, 1, 2, \ldots$,

$$\left|\sum_{n=0}^{m} b_n\omega_n^{(\alpha,\beta)}P_n^{(\alpha,\beta)}(x)\right| \leqq K\left\{(1+x)\sum_{n=0}^{m}(n+1)^{\beta+2}b_n + \sum_{n=0}^{m}(n+1)^{\beta}b_n\right\}. \tag{6.19}$$

Now we put $-1<x<0$, i.e., $x=\cos\theta$, $\pi/2<\theta<\pi$. Then, by (6.19), we have, for $0\leqq m\leqq[c/(\pi-\theta)]$,

$$\begin{aligned} \left|\sum_{n=0}^{m} b_n\omega_n^{(\alpha,\beta)}P_n^{(\alpha,\beta)}(\cos\theta)\right| &\leqq K\left\{\frac{(\pi-\theta)^2}{2}\sum_{n=0}^{[c/(\pi-\theta)]}(n+1)^{\beta+2}b_n + \sum_{n=0}^{[c/(\pi-\theta)]}(n+1)^{\beta}b_n\right\} \leqq \\ &\leqq K_1\sum_{n=0}^{[c/(\pi-\theta)]}(n+1)^{\beta}b_n \leqq K_2G(\theta) \qquad \left(\frac{\pi}{2}<\theta<\pi\right). \end{aligned} \tag{6.20}$$

On the other hand, we can prove (6.3) by the same method of estimation as in (6.13). By (6.20) and (6.3), we have the case $\pi/2<\theta<\pi$ of (6.1). When we put $x=-1$ in (6.19), we have, for $m=0, 1, 2, \ldots$,

$$\left|\sum_{n=0}^{m} b_n\omega_n^{(\alpha,\beta)}P_n^{(\alpha,\beta)}(-1)\right| \leqq K\sum_{n=0}^{m}(n+1)^{\beta}b_n \leqq KG(\pi).$$

Hence we get the case $\theta=\pi$ of (6.1). Thus we have the case $\pi/2<\theta\leqq\pi$ of (6.1).

We have proved (6.1). The inequality (6.4) is trivial from (6.1). Thus Lemma 8 is proved.

Proof of Theorem 5. By the first part of Lemma 8, Jacobi series (1.5) converges to a function $g(\theta)$ in $0<\theta<\pi$. Since $f(\theta)G(\theta)\in L([0,\pi];\alpha,\beta)$, so does $f(\theta)g(\theta)$ by (6.4). We have, for $m=0, 1, 2, \ldots$,

$$\sum_{n=0}^{m} a_nb_n\omega_n^{(\alpha,\beta)} = \int_0^{\pi} f(\theta)\left(\sum_{n=0}^{m} b_n\omega_n^{(\alpha,\beta)}P_n^{(\alpha,\beta)}(\cos\theta)\right)\varrho^{(\alpha,\beta)}(\theta)\,d\theta = \int_0^{\pi/2} + \int_{\pi/2}^{\pi} = W_1+W_2 \tag{6.21}$$

say. Further we put

$$(6.22)\quad W_1 = \left(\int_0^{c/m} + \int_{c/m}^{\pi/2}\right) f(\theta)\left(\sum_{n=0}^{m} b_n \omega_n^{(\alpha,\beta)} P_n^{(\alpha,\beta)}(\cos\theta)\right)\varrho^{(\alpha,\beta)}(\theta)\,d\theta = W_{1,1} + W_{2,2},$$

where c is a positive constant in (2.6). By (6.1) and assumption, we get

$$(6.23)\qquad |W_{1,1}| \leqq K\int_0^{c/m} |f(\theta)|\,G(\theta)\varrho^{(\alpha,\beta)}(\theta)\,d\theta \to 0 \quad\text{as}\quad m\to\infty.$$

We write

$$W_{1,2} = \int_{c/m}^{\pi/2} f(\theta)g(\theta)\varrho^{(\alpha,\beta)}(\theta)\,d\theta - \int_{c/m}^{\pi/2} f(\theta)\left(\sum_{n=m+1}^{\infty} b_n\omega_n^{(\alpha,\beta)} P_n^{(\alpha,\beta)}(\cos\theta)\right)\varrho^{(\alpha,\beta)}(\theta)\,d\theta =$$

$$(6.24)\qquad = Z_1 - Z_2.$$

Since $f(\theta)g(\theta)\in L([0,\pi];\alpha,\beta)$, we have

$$(6.25)\qquad Z_1 \to \int_0^{\pi/2} f(\theta)g(\theta)\varrho^{(\alpha,\beta)}(\theta)\,d\theta \quad\text{as}\quad m\to\infty.$$

Let $y_1 = 1/2$ and, for $n \geqq 2$,

$$y_n = n^{1/(2\alpha+3)} \quad\text{if}\quad b_n \leqq Kn^{-3/2},\quad\text{and}\quad y_n = \left(\frac{K}{n^{1/2}b_n}\right)^{1/(2\alpha+3)} \quad\text{if}\quad b_n > Kn^{-3/2}.$$

Then

$$(6.26)\qquad n > y_n > 0,\quad y_n\to\infty \quad\text{and}\quad (n^{1/2}b_n)y_n^{\alpha+3/2}\to 0 \quad\text{as}\quad n\to\infty.$$

We put

$$(6.27)\quad Z_2 = \left(\int_{c/m}^{c/y_m} + \int_{c/y_m}^{\pi/2}\right) f(\theta)\left(\sum_{n=m+1}^{\infty} b_n\omega_n^{(\alpha,\beta)} P_n^{(\alpha,\beta)}(\cos\theta)\right)\varrho^{(\alpha,\beta)}(\theta)\,d\theta = Z_{2,1} + Z_{2,2}.$$

Hence, by (6.27), (6.26) and (6.2) (let $N\to\infty$ in (6.2)),

$$(6.28)\qquad |Z_{2,1}| \leqq K\int_{c/m}^{c/y_m} |f(\theta)|\,G(\theta)\varrho^{(\alpha,\beta)}(\theta)\,d\theta \to 0 \quad\text{as}\quad m\to\infty.$$

Further, by (6.27), (6.2), (J6) and (6.26) (let $N\to\infty$ in (6.2)),

$$(6.29)\qquad \begin{aligned} |Z_{2,2}| &\leqq K\int_{c/y_m}^{\pi/2} |f(\theta)|\,\theta^{-\alpha-3/2}(m+1)^{1/2} b_{m+1}\varrho^{(\alpha,\beta)}(\theta)\,d\theta \leqq \\ &\leqq K_1 y_m^{\alpha+3/2}(m^{1/2}b_m)\int_{c/y_m}^{\pi/2} |f(\theta)|\varrho^{(\alpha,\beta)}(\theta)\,d\theta \to 0 \quad\text{as}\quad m\to\infty. \end{aligned}$$

By (6.27), (6.28) and (6.29), we have

$$(6.30)\qquad Z_2 \to 0 \quad\text{as}\quad m\to\infty.$$

From (6.24), (6.25) and (6.30), we get

$$W_{1,2} \to \int_0^{\pi/2} f(\theta)g(\theta)\varrho^{(\alpha,\beta)}(\theta)\,d\theta \quad \text{as} \quad m\to\infty. \tag{6.31}$$

Thus, by (6.31), (6.23) and (6.22),

$$W_1 \to \int_0^{\pi/2} f(\theta)g(\theta)\varrho^{(\alpha,\beta)}(\theta)\,d\theta \quad \text{as} \quad m\to\infty. \tag{6.32}$$

When we consider W_2, we define y_n as follows:

(I) if $\beta\geqq 1/2$, let $y_1=1/2$ and, for $n\geqq 2$,

$$y_n = n^{1/(2\beta+1)} \quad \text{if} \quad b_n\leqq Kn^{-3/2}, \quad \text{and} \quad y_n=\left(\frac{K}{n^{1/2}b_n}\right)^{1/2(\beta+1)} \quad \text{if} \quad b_n > Kn^{-3/2},$$

(II) if $-1/2\leqq\beta<1/2$, let $y_1=1/2$ and, for $n\geqq 2$,

$$y_n = n^{1/2} \quad \text{if} \quad b_n\leqq Kn^{-3/2}, \quad \text{and} \quad y_n=\left(\frac{K}{n^{1/2}b_n}\right)^{1/2} \quad \text{if} \quad b_n > Kn^{-3/2}.$$

From (I) or (II), we have that $n>y_n>0$, $y_n\to\infty$ and $(n^{1/2}b_n)y_n^{\beta+1/2}\to 0$ as $n\to\infty$. Now we shall obtain

$$W_2 \to \int_{\pi/2}^{\pi} f(\theta)g(\theta)\varrho^{(\alpha,\beta)}(\theta)\,d\theta \quad \text{as} \quad m\to\infty$$

by the same method as in (6.32). Hence, combining this with (6.32) and (6.21),

$$\sum_{n=0}^{\infty} a_n b_n \omega_n^{(\alpha,\beta)} = \int_0^{\pi} f(\theta)g(\theta)\varrho^{(\alpha,\beta)}(\theta)\,d\theta.$$

Thus Theorem 5 is proved.

Proof of Corollary 2. Since $\beta\geqq\alpha>-1/2$ and $0<\delta<\alpha+1/2$, we have $-\delta+\alpha+1/2>0$ and $-\delta+\beta-1/2>-1$. If we put $b_0=0$ and $b_n=n^{-\delta-1/2}\varphi(n)$ for $n=1,2,\dots$, then $\{b_n\}$ satisfies (J5), (J6) and (J7). We get

$$\sum_{n=0}^{[1/\theta]}(n+1)^{\alpha+1}b_n \leqq K\theta^{\delta-\alpha-3/2}\varphi\left(\left[\frac{1}{\theta}\right]\right) \quad \text{for} \quad 0\leqq\theta\leqq\frac{\pi}{2},$$

$$\sum_{n=0}^{[1/(\pi-\theta)]}(n+1)^{\beta}b_n \leqq K_1(\pi-\theta)^{\delta-\beta-1/2}\varphi\left(\left[\frac{1}{\pi-\theta}\right]\right) \quad \text{for} \quad \frac{\pi}{2}<\theta\leqq\pi.$$

Hence, from (1.8) and (S4),

$$G(\theta)\leqq K_2\theta^{\delta-\alpha-3/2}(\pi-\theta)^{\delta-\beta-1/2}\varphi\left(\frac{1}{\theta(\pi-\theta)}\right).$$

Since

$$f(\theta)\theta^{\delta-\alpha-3/2}(\pi-\theta)^{\delta-\beta-1/2}\varphi\left(\frac{1}{\theta(\pi-\theta)}\right)\in L([0,\pi];\alpha,\beta),$$

so does $f(\theta)G(\theta)$. Thus, from Theorem 5, the series $\sum_{n=1}^{\infty} n^{-\delta+1/2}\varphi(n)\,a_n\omega_n^{(\alpha,\beta)}$ converges. From (2.2), we have

$$n=\frac{1}{2}\omega_n^{(\alpha,\beta)}\left\{1+\frac{B^*}{n}+O\left(\frac{1}{n^2}\right)\right\},$$

where B^* is a constant depending only on α and β. Hence $\sum_{n=1}^{\infty} n^{-\delta+1/2}\varphi(n)\,a_n$ converges. Thus Corollary 2 is proved.

Proof of Corollary 3. Since $\alpha\geqq\beta\geqq-1/2$ and $\alpha+1/2<\delta<\alpha+3/2$, we have $-1<-\delta+\alpha+1/2<0$ and $-\delta+\beta-1/2<1$. In Theorem 5 we put $b_0=0$ and $b_n=n^{-\delta-1/2}\varphi(n)$ for $n=1, 2, \ldots$. Then it is sufficient to notice that

$$\sum_{n=0}^{[1/(\pi-\theta)]}(n+1)^{\beta}b_n\leqq K \qquad \text{for} \quad \frac{\pi}{2}<\theta\leqq\pi.$$

The rest of the proof is similar to that of Corollary 3.

References

[1] S. Aljančić, R. Bojanić and M. Tomić, Sur la valuer asymptotique d'une classe des intégrales définies, *Publ. Inst. Math. Acad. Serbe Sci.*, 7 (1954), 81—84.

[2] ——— Sur l'intégrabilité de certaines séries trigonométriques, *Publ. Inst. Math. Acad. Serbe Sci.*, **8** (1955), 67—84.

[3] R. Askey, A transplantation theorem for Jacobi series, *Illinois J. Math.*, **13** (1969), 583—590.

[4] N. K. Bary, *A treatise on trigonometric series*. Vol. II, Pergamon Press (1964).

[5] H. Bavinck, *A special class of Jacobi series and some applications*, Math. Centrum Amsterdam, report TW 116 (1970).

[6] C. C. Ganser, Integrability of ultraspherical series, *Duke Math. J.*, **33** (1966), 539—545.

[7] S. Igari, Some integrability theorems of trigonometric series and monotone decreasing functions, *Tôhoku Math. J.*, (2) **12** (1960), 139—146.

[8] M. Izumi and S. Izumi, Integrability theorem for Fourier series and Parseval equation, *J. Math. Anal. Appl.*, **18** (1967), 252—261.

[9] Y. L. Luke, *The special functions and their approximations*. Vol. I, Academic Press (1969).

[10] L. McFadden, Absolute Nörlund summability, *Duke Math. J.*, **9** (1942), 168—207.

[11] G. Szegő, *Orthogonal polynomials*, Amer. Math. Soc. (1967).

DEPARTMENT OF MATHEMATICS
FACULTY OF GENERAL EDUCATION
HIROSAKI UNIVERSITY, JAPAN

(Received June 26, 1973)